Indian Statistical Institute
M. Math. I year II Semester
Mid-Semestral Examination 2003-2004

Time: 3 hours Algebra II Date: 04-05-2004

Answer all questions.

1. (a) Define the localization of a commutative ring R with respect to a multiplicative set.
(b) If A is an ideal of R, show that $1 + A = \{1 + a : a \in A\}$ is a multiplicative set.
(c) Defining all the concepts you use, show that, if A is an ideal of R, then there is a bijection between the prime ideals of $R_{(1+A)}$ and the set of prime ideals P of R such that $P + A \neq R$. [3+3+6]

2. Define a local ring. Give an example. Show that the number of elements in a minimal generating set of an R-module is not unique for a commutative ring R in general, but it is well-defined for local rings.
[2+4+2+6]

3. Determine the set of all invertible elements in the following rings:
(a) $k[X]$; (b) $k[[X]]$,
where k is a field. [4+6]

4. Prove that the polynomial ring $k[X_1, X_2]$, k a field, is a unique factorization domain, but not a prinicipal ideal domain. [8+4]

5. State Eisenstein's criteria for the irreducibility of a polynomial over a principal ideal domain.
Determine the dimension of the splitting field, over Q, for each of the polynomials below:
(i) $X^6 + X^3 + 1$ over Q
(ii) $X^{p-1} + X^{p-2} + \ldots + X + 1$ over Q, p - a prime. [3+4+3]

6. Define the characteristic of a field. Find all fields k such that the map $\theta : k \to k$ such that $\theta(x) = x^5$ is a field automorphism of k. [2+8]

7. Show that a finite, multiplicatively closed, subset of a field is a cyclic group. [8]